Counting

Example

List all different results you can get from flipping a coin 4 times

There are $2^4 = 16$ *outcomes:*

$$HHHH, HHHT, HHTH, HHTT, HTHH, HTHT, HTTH, HTTT,$$
$$THHH, THHT, THTH, THTT, TTHH, TTHT, TTTH, TTTT$$

Example

How many ways are there to arrange the letters of *GHOT*?

$4! = 4 \times 3 \times 2 \times 1 = 24$ *ways.*

Example

If we put all our 4 coin-flips in order and converted them to binary, what would we see?

Writing $H = 0$ *and* $T = 1$*:*

$$HHHH = 0000_2 = 0, \quad HHHT = 0001_2 = 1, \quad HHTH = 0010_2 = 2, \quad \dots \quad TTTT = 1111_2 = 15$$

We see the numbers 0 to 15 in binary.

Example

If I listed all possible flips of a coin 10 times in alphabetical order, ie

$$HHHHHHHHHH$$
$$HHHHHHHHHT$$
$$HHHHHHHHTH$$
$$\vdots$$
$$HTHTHTHTHT$$
$$\vdots$$

what would appear 10^{th} in this sequence? 100^{th}?

The n^{th} *item corresponds to* $(n-1)$ *in binary (padded to 10 digits), with* $H = 0$, $T = 1$.
10^{th}: $9 = 1001_2 = 0000001001_2 \Rightarrow HHHHHHTHHT$
100^{th}: $99 = 1100011_2 = 0001100011_2 \Rightarrow HHHTTHHHTT$

Arrangements and Permutations

Example (a) How many anagrams of $FRIDAY$ are there?

(b) How many words are there using 4 *different* letters of $FRIDAY$?

(c) How many words are there using 4 *not necessarily distinct* letters of $FRIDAY$?

(d) How many **permutations** of $FRIDAY$ have the two vowels together?

(a) $6! = 720$

(b) Choose 4 letters from 6, then arrange: $\frac{6!}{(6-4)!} = {}^6P_4 = 6 \times 5 \times 4 \times 3 = 360$

(c) Each of 4 positions can be any of 6 letters: $6^4 = 1296$

(d) Treat the vowels I and A as a single unit. We have 5 units to arrange: $5! = 120$. *The vowels can be in either order (IA or AI):* $\times 2$. *Total:* $5! \times 2 = 240$

Example

- In how many ways can 15 knights sit around a round table?
- In how many ways can 7 different coloured beads be threaded onto a necklace?

- *For circular arrangements, fix one person and arrange the rest:* $(15-1)! = 14!$
- *A necklace can be flipped (reflections are the same), so:* $\frac{(7-1)!}{2} = \frac{6!}{2} = 360$

Example
In how many ways can eight rooks be placed on a chessboard so that no two are on the same row or column?

Place rooks row by row. Row 1: 8 choices. Row 2: 7 remaining columns. Row 3: 6 columns, etc.

$$8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 8! = 40320$$

Example
How many numbers divisible by 6 can be made by arranging 4 different digits from $1, 2, 3, 4$ and 5

Divisible by 6 means divisible by both 2 and 3.
Divisible by 2: *Must end in 2 or 4.*
Divisible by 3: *Digit sum must be divisible by 3. Sum of all five digits:* $1+2+3+4+5 = 15$.
If we use 4 digits, we omit one. For the sum to be divisible by 3, we must omit a digit $\equiv 0 \pmod 3$*, i.e., omit 3.*
Using digits $\{1, 2, 4, 5\}$ *(sum = 12, divisible by 3):*

- *End in 2: arrange* $\{1, 4, 5\}$ *in first 3 positions:* $3! = 6$ *ways*
- *End in 4: arrange* $\{1, 2, 5\}$ *in first 3 positions:* $3! = 6$ *ways*

Total: $6 + 6 = 12$

Example
In how many ways can four married couples be arranged around a round table so that every couple sits together?

Treat each couple as a single unit. Arrange 4 units around a round table: $(4-1)! = 3! = 6$ *ways.*
Within each couple, they can swap seats: $2^4 = 16$ *ways.*
Total: $6 \times 16 = 96$

Example (a) How many permutations are there of *aaabbbbcc*?

(b) What is the coefficient of $a^3b^4c^2$ in the expansion of $(a+b+c)^9$?

(a) $\dfrac{9!}{3! \cdot 4! \cdot 2!} = \dfrac{362880}{6 \cdot 24 \cdot 2} = \dfrac{362880}{288} = 1260$

(b) By the multinomial theorem, this is $\dbinom{9}{3, 4, 2} = \dfrac{9!}{3! \cdot 4! \cdot 2!} = 1260$

(This is the same calculation - each permutation of aaabbbbcc corresponds to a way of choosing which positions get a, b, c.)

Combinations

Example

In how many ways can a quiz team of six people be selected from a group of twelve people?

$$\binom{12}{6} = \frac{12!}{6!\cdot 6!} = \frac{12\times 11\times 10\times 9\times 8\times 7}{6!} = 924$$

Example

In how many ways can a set 0 of two people, set 1A of fourteen people and set 1B of fourteen people be chosen from a group of 30 people?

Choose 2 for set 0, then 14 for set 1A from the remaining 28, then the last 14 go to set 1B:

$$\binom{30}{2}\times\binom{28}{14}\times\binom{14}{14} = 435\times 40116600\times 1 = 17,450,721,000$$

Example

In the national lottery a player buys a ticket from a machine which lists six out of the numbers 1 to 49 inclusive. A machine selects six balls, without replacement, from a set of forty-nine balls which are numbered 1 to 49,

(a) How many different ways are there of choosing the six balls?

(b) In how many ways can a player match exactly three of the six winning numbers?

(c) In how many ways can a player match three or more of the six winning numbers?

(d) In addition, the machine draws a seventh ball, which is called the bonus ball. In how many ways can a player match five numbers and the bonus ball?

(a) $\binom{49}{6} = 13,983,816$

(b) Choose 3 from the 6 winning numbers, and 3 from the 43 losing numbers:

$$\binom{6}{3}\times\binom{43}{3} = 20\times 12341 = 246,820$$

(c) Match exactly k numbers: $\binom{6}{k}\times\binom{43}{6-k}$

$$\begin{aligned} k=3: &\quad 20\times 12341 = 246820 \\ k=4: &\quad 15\times 903 = 13545 \\ k=5: &\quad 6\times 43 = 258 \end{aligned}$$

$$k = 6: \quad 1 \times 1 = 1$$
$$\textit{Total:} \quad 260,624$$

(d) Match 5 of 6 winning numbers, miss 1 winning number (which becomes the bonus ball), and your 6th number is the bonus ball. There are $\binom{6}{5} = 6$ *ways to choose which 5 winning numbers you match.*

Example

In how many ways can a committee consisting of five women and six men be formed from eight women and ten men?

Choose 5 women from 8 and 6 men from 10:

$$\binom{8}{5} \times \binom{10}{6} = 56 \times 210 = 11{,}760$$

Example

Prove that

$$\binom{k}{k} + \binom{k+1}{k} + \binom{k+2}{k} + \cdots + \binom{n}{k} = \binom{n+1}{k+1}$$

Combinatorial proof: *The RHS counts ways to choose $k+1$ items from $n+1$ items labelled $0, 1, 2, \ldots, n$.*

Partition by the largest item chosen. If the largest item is j (where $k \leq j \leq n$), we must choose the remaining k items from $\{0, 1, \ldots, j-1\}$, which can be done in $\binom{j}{k}$ ways.

Summing over all possible largest items: $\displaystyle\sum_{j=k}^{n} \binom{j}{k} = \binom{n+1}{k+1}$ *(This is the "hockey stick identity".)*

Example

Suppose that a, b, c, d, e are positive integers. How many solutions to the equation $a + b + c + d + e = 14$ are there?

Substitute $a' = a - 1, b' = b - 1$, etc., where $a', b', c', d', e' \geq 0$.

Then $a' + b' + c' + d' + e' = 14 - 5 = 9$.

By stars and bars: $\binom{9+5-1}{5-1} = \binom{13}{4} = 715$

Example

What if a, b, c, d, e are *non-negative* integers. How many solutions to the equation $a + b + c + d + e = 14$ are there?

By stars and bars directly: $\binom{14+5-1}{5-1} = \binom{18}{4} = 3060$

Example (Poker)

What is the probability of getting:

(a) 1 pair
(b) 2 pair
(c) 3 pair
(d) 3 of a kind
(e) 2 sets of 3 of a kind
(f) 4 of a kind
(g) 1 pair and 3 of a kind
(h) 1 pair and 4 of a kind
(i) Straight
(j) Flush
(k) Straight Flush
(l) No pair

In poker with 5 cards and 6 cards

Total 5-card hands: $\binom{52}{5} = 2,598,960$. *For 5-card poker:*

(a) ***1 pair:*** *Choose rank for pair: 13. Choose 2 suits:* $\binom{4}{2} = 6$. *Choose 3 other ranks:* $\binom{12}{3}$. *Choose suit for each:* 4^3. *Total:* $13 \times 6 \times 220 \times 64 = 1,098,240$. *Probability* $\approx 42.3\%$

(b) ***2 pair:*** $\binom{13}{2} \times \binom{4}{2}^2 \times 11 \times 4 = 123,552$. *Probability* $\approx 4.75\%$

(c) ***3 pair:*** *Impossible with 5 cards (would need 6 cards)*

(d) ***3 of a kind:*** $13 \times \binom{4}{3} \times \binom{12}{2} \times 4^2 = 54,912$. *Probability* $\approx 2.11\%$

(e) ***Full house (pair + 3 of a kind):*** $13 \times \binom{4}{3} \times 12 \times \binom{4}{2} = 3,744$. *Probability* $\approx 0.14\%$

(f) ***4 of a kind:*** $13 \times 1 \times 12 \times 4 = 624$. *Probability* $\approx 0.024\%$

(i) ***Straight:*** *10 possible straights (A-5 through 10-A)* $\times 4^5 - 40$ *(exclude straight flushes)* $= 10,200$. *Probability* $\approx 0.39\%$

(j) ***Flush:*** $4 \times \binom{13}{5} - 40 = 5,108$. *Probability* $\approx 0.20\%$

(k) ***Straight flush:*** $10 \times 4 = 40$. *Probability* $\approx 0.0015\%$

(l) ***No pair (high card):*** $\binom{13}{5} \times 4^5 - 10,200 - 5,108 - 40 = 1,302,540$. *Probability* $\approx 50.1\%$

Example

Calculate $\binom{n}{0}+\binom{n}{1}+\binom{n}{2}+\cdots+\binom{n}{n}$

By the binomial theorem with $x=1$:

$$(1+1)^n=\sum_{k=0}^{n}\binom{n}{k}1^k\cdot 1^{n-k}=\sum_{k=0}^{n}\binom{n}{k}=2^n$$

Combinatorially: this counts all subsets of an n-element set.

Example

Calculate $\binom{n}{0}-\binom{n}{1}+\binom{n}{2}-\cdots+(-1)^n\binom{n}{n}$

By the binomial theorem with $x=-1$:

$$(1-1)^n=\sum_{k=0}^{n}\binom{n}{k}(-1)^k=0 \quad (\text{for } n\geq 1)$$

This shows there are equal numbers of subsets of even and odd size.

Example

Calculate $1+2+3+\cdots+n$

$$\sum_{k=1}^{n}k=\binom{n+1}{2}=\frac{n(n+1)}{2}$$

Combinatorially: $k=\binom{k}{1}$ counts ways to choose 1 item from k items. By the hockey stick identity, the sum equals $\binom{n+1}{2}$.

Example

Calculate $1^2+2^2+3^2+\cdots+n^2$

Note that $k^2=2\binom{k}{2}+\binom{k}{1}=k(k-1)+k$.

$$\begin{aligned}\sum_{k=1}^{n}k^2&=2\sum_{k=1}^{n}\binom{k}{2}+\sum_{k=1}^{n}\binom{k}{1}\\&=2\binom{n+1}{3}+\binom{n+1}{2} \quad (\text{hockey stick})\\&=2\cdot\frac{(n+1)n(n-1)}{6}+\frac{(n+1)n}{2}\end{aligned}$$

$$= \frac{(n+1)n(n-1)}{3} + \frac{(n+1)n}{2}$$
$$= \frac{(n+1)n}{6}\left(2(n-1)+3\right) = \frac{n(n+1)(2n+1)}{6}$$

Partitions

Example

How many ordered partitions of 10 are there? [Ordered partitions are partitions where the order of the summands matters, eg $3 = 3 = 2+1 = 1+2 = 1+1+1$]

1. How many ordered partitions of n are there?
2. How many *terms* are there in the ordered partitions of n?

1. *Consider n as a row of n ones: $1\ 1\ 1\ \cdots\ 1$. An ordered partition corresponds to choosing where to place "+" signs in the $n-1$ gaps between consecutive ones.*

   Each gap either has a "+" or doesn't: 2^{n-1} ordered partitions.

   For $n = 10$: $2^9 = 512$ ordered partitions.

2. *Each gap contributes a term boundary or not. A partition with k terms has $k-1$ plus signs. The number of partitions with k terms is $\binom{n-1}{k-1}$.*

   Total terms: $\sum_{k=1}^{n} k\binom{n-1}{k-1} = \sum_{j=0}^{n-1}(j+1)\binom{n-1}{j} = \sum_{j=0}^{n-1} j\binom{n-1}{j} + 2^{n-1}$

   $= (n-1)2^{n-2} + 2^{n-1} = (n+1)2^{n-2}$

Example

How many *unordered* partitions of 5 are there?

List them systematically by largest part:

$$5 = 5$$
$$5 = 4+1$$
$$5 = 3+2$$
$$5 = 3+1+1$$
$$5 = 2+2+1$$
$$5 = 2+1+1+1$$
$$5 = 1+1+1+1+1$$

There are $p(5) = 7$ unordered partitions.

Example

Show the number of partitions of n into at most k parts is the same as the number of partitions of $n+k$ into exactly k parts

Given a partition of n into at most k parts, pad it with zeros to have exactly k parts, then add 1 to each part. This gives a partition of $n+k$ into exactly k positive parts.

This map is reversible: given a partition of $n+k$ into exactly k parts, subtract 1 from each part to get a partition of n into at most k parts (some parts may become zero).

Example: $n = 5, k = 3$. Partition $5 = 3 + 2$ (2 parts) $\longrightarrow (3,2,0) \longrightarrow (4,3,1)$, a partition of 8 into 3 parts.

Example

Show that the number of partitions of n into self-conjugate parts is equal to the number of partitions with *distinct* odd parts

A self-conjugate partition has a symmetric Ferrers diagram. Each such partition can be decomposed into "hooks" centred on the diagonal. Each hook has odd size $2m+1$ (one diagonal cell plus m cells in each direction).

These hooks have distinct odd sizes, giving a bijection with partitions into distinct odd parts.

Example: The self-conjugate partition $(4,2,1,1)$ of 8 has hooks of sizes 7 and 1, corresponding to $8 = 7 + 1$.

Example

Show that the number of partitions of n into even parts is equal to the number of partitions of n in which each part appears an even number of times

Given a partition into even parts, write each part $2m$ as $m+m$ (two copies of m). This doubles the number of each part size, giving a partition where each part appears an even number of times.

Conversely, given a partition where each part appears an even number of times, pair up identical parts: two copies of m become one copy of $2m$.

Example: $8 = 4 + 2 + 2 \leftrightarrow 8 = 2 + 2 + 1 + 1 + 1 + 1$

Example

Show that the number of partitions of n into an even number of parts parts minus the number of partitions of n into an odd number of parts parts can be $-1, 0, 1$

*This is **Euler's pentagonal number theorem**. The difference equals $(-1)^k$ if $n = \frac{k(3k\pm1)}{2}$ (a pentagonal number), and 0 otherwise.*

The proof uses a sign-reversing involution: for most partitions, we can pair a partition with an even number of parts with one having an odd number of parts, so they cancel. Only partitions of pentagonal numbers fail to pair up.

Pentagonal numbers: $1, 2, 5, 7, 12, 15, 22, 26, \ldots$